Local unitary equivalent consistence for n-party states and their $n-1$-party reduced density matrices

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We present that the local unitary equivalence of n-party pure states is consistent with the one of their $(n-1)$-party reduced density matrices. As an application, we obtain the local invariants for a class of tripartite pure qudits.

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As the properties of entanglement for n-party quantum states remain invariant under local unitary transformations on subsystems, the entanglement can be characterized in principle by all invariants under local transformations [1]. The polynomial invariants of local unitary transformations have been discussed [2–4]. For two bipartite pure states the set of Schmidt coefficients forms a complete set of invariants under local unitary transformations [5]. For multipartite pure and mixed states, complete sets of invariants have not been found except some particular cases, e.g. two-qubit mixed states [6], three-qubit pure states [7, 8] and generic mixed states [9–11]. S. Albeverio *et.al.* [12] introduced the notion of generic states and obtained criteria for local equivalence of such states. In [13] they further introduce the concept of CHG (commuting high generic) states maintaining the criteria of local equivalence and characterize the equivalence classes under local unitary transformations for the set of tripartite states whose partial trace with respect to one of the subsystems belongs to the class of CHG mixed states. In this letter we show that the equivalent problem under local unitary transformations for any n-party pure state can be reduced to the problem for its $(n-1)$-party reduced states. According to the result, one can easily obtain the set of invariants for n-party pure states with the known invariants for $(n-1)$-party mixed states.

Let W_j be a $d_j \times d_j$ unitary matrix, and the subsystem $\mathcal{H}_{A_j}$ be a Hilbert space of d_j dimension. We first give a lemma which can be proved as in [14] (p.111, Exercise 2.80).

Lemma If two n-party pure states $|\psi\rangle$ and $|\psi'\rangle$ acting on the system $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$ have same $(n-1)$-party reduced density matrices, then for each party $j \in$

$\{1,\ 2,\ \cdots,\ n\}$, there is a unitary transformation W_j such that $|\psi'\rangle = (I \otimes \cdots \otimes I \otimes W_j \otimes I \otimes \cdots \otimes I)|\psi\rangle$, where W_j acts on the subsystem of party j.

Remark 1 The lemma shows that two pure states with the same reduced matrices are equivalent under local unitary transformations. However, it is not true for mixed states. For example, let

$$\rho_1 = \frac{1}{3}|\psi^+\rangle\langle\psi^+| + \frac{2}{3}|\psi^-\rangle\langle\psi^-|, \tag{1}$$

$$\rho_2 = \frac{1}{2}|\psi^+\rangle\langle\psi^+| + \frac{1}{2}|\psi^-\rangle\langle\psi^-|, \tag{2}$$

where

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle).$$

It is obvious that $\mathrm{Tr}_{A_1}(\rho_1) = \mathrm{Tr}_{A_1}(\rho_2)$, $\mathrm{Tr}_{A_2}(\rho_1) = \mathrm{Tr}_{A_2}(\rho_2)$ and $\mathrm{Tr}_{A_3}(\rho_1) = \mathrm{Tr}_{A_3}(\rho_2)$. However, there is no local unitary transformation W_j such that $\rho_1 = (I \otimes \cdots \otimes I \otimes W_j \otimes I \otimes \cdots \otimes I)\rho_2(I \otimes \cdots \otimes I \otimes W_j \otimes I \otimes \cdots \otimes I)^{\dagger}$ for any party j as the ranks of ρ_1 and ρ_2 are different.

One may ask that if the reduced matrices of two pure multipartite states are equivalent under local unitary transformations, whether the two pure multipartite states are equivalent under local unitary transformations? The answer is affirmative:

Proposition If one of $(n-1)$-party reduced density matrices for n-party pure states $|\psi\rangle$ is local unitary equivalent to the corresponding $(n-1)$-party reduced density matrix for n-party pure states $|\psi'\rangle$ in the system $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$, the two n-party pure states $|\psi\rangle$ and $|\psi'\rangle$ are also local unitary equivalent.

Proof. Let $j \in \{1,\ 2,\ \cdots,\ n\}$ be a party label. Set $\rho_{(j)}$ and $\rho'_{(j)}$ be the $(n-1)$-party reduced density matrix obtained by taking the partial trace of $|\psi\rangle$ and $|\psi'\rangle$ over party j, respectively. In addition, $\rho_{(j)}$ and $\rho'_{(j)}$ are equivalent under local unitary transformations. Hence there exist unitary operators $U_1,\ \cdots, U_{j-1},\ U_{j+1},\ \cdots,\ U_n$ such that $\rho'_{(j)} = (U_1 \otimes \cdots \otimes U_{j-1} \otimes U_{j+1} \otimes \cdots \otimes U_n)\rho_{(j)}(U_1 \otimes \cdots \otimes U_{j-1} \otimes U_{j+1} \otimes \cdots \otimes U_n)^{\dagger}$.

Suppose U_j be a unitary operator in the subsystem $\mathcal{H}_{A_j}$. Let $|\phi\rangle = (U_1 \otimes U_2 \otimes \cdots \otimes U_n)|\psi\rangle$,

then

$$\begin{aligned}\sigma_{(j)} &= \mathrm{Tr}_j(|\phi\rangle\langle\phi|) \\ &= (U_1 \otimes \cdots \otimes U_{j-1} \otimes U_{j+1} \otimes \cdots \otimes U_n)\rho_{(j)} \\ &\quad\ (U_1 \otimes \cdots \otimes U_{j-1} \otimes U_{j+1} \otimes \cdots \otimes U_n)^\dagger \\ &= \rho'_{(j)},\end{aligned}$$

where $\sigma_{(j)}$ is the $(n-1)$-party reduced density matrix of n-party pure states $|\phi\rangle$ for $n-1$ subsystems except $\mathcal{H}_{A_j}$. As the pure state $|\phi\rangle$ have same reduced density matrices as $|\psi'\rangle$, according to the lemma, we obtain that there exists a unitary transformation V_j such that $|\phi\rangle = (I \otimes \cdots \otimes I \otimes V_j \otimes I \otimes \cdots \otimes I)|\psi'\rangle$, where the unitary operator V_j acts on the subsystem $\mathcal{H}_{A_j}$, i.e., $(U_1 \otimes U_2 \otimes \cdots \otimes U_n)|\psi\rangle = (I \otimes \cdots \otimes I \otimes V_j \otimes I \otimes \cdots \otimes I)|\psi'\rangle$. Thus $|\psi'\rangle = (U_1 \otimes U_2 \otimes \cdots \otimes V_j^\dagger U_j \otimes \cdots \otimes U_n)|\psi\rangle$, which proves the result. $\square$

Remark 2 If $|\psi\rangle$ and $|\psi'\rangle$ only have one same $(n-1)$-party reduced density matrix, e.g., $\rho_{(1)} = \rho'_{(1)}$, and $\rho_{(i)} \neq \rho'_{(i)}$ for any $i = 2,\ 3,\ \cdots,\ n$, then n-partite pure states $|\psi\rangle$ and $|\psi'\rangle$ are still equivalent under unitary transformations.

If two n-party pure states $|\psi\rangle$ and $|\psi'\rangle$ in the systems $\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}\otimes\cdots\otimes\mathcal{H}_{A_n}$ are equivalent under local unitary transformations, then their reduced density matrices must be equivalent under local unitary transformations. Thus the local unitary equivalence of multipartite pure states is consistent with the local unitary equivalence of their reduced density matrices. This is, however, not the case for multipartite mixed states, e.g. the states in (1) and (2).

The proposition yields the fact that the problem of invariants for n-party pure states can be reduced to the one for $(n-1)$-party mixed states. For example, two pure states are equivalent under local unitary transformations if and only if they have the same values of the invariants $I_\alpha,\ \alpha = 1,\ \cdots,\ n$ [15].

As an application, we discuss the invariants for tripartite pure qudits in the following. Let $\mathcal{H}_i (i = 1,\ 2,\ 3)$ be complex Hilbert spaces of dimension d. A mixed state ρ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ with $\mathrm{rank}(\rho) = n \le d^2$ can be decomposed according to its eigenvalues λ_i and eigenvectors $|v_i\rangle,\ i = 1,\ 2,\ \cdots,\ n$:

$$\rho = \sum_{i=1}^{n} \lambda_i |v_i\rangle\langle v_i|,$$

where $|v_i\rangle$ has the form

$$|v_i\rangle=\sum_{k,l=1}^{n}a_{kl}^{i}|kl\rangle,\ a_{kl}^{i}\in\mathcal{C},\ \sum_{k,l=1}^{n}a_{kl}^{i}a_{kl}^{i*}=1,\ i=1,\ 2,\ \cdots,\ n.$$

Let A_i denote the matrix given by $(A_i)_{kl}=a_{kl}^{i}$, $\rho_i=\mathrm{Tr}_2|v_i\rangle\langle v_i|=A_iA_i^{\dagger}$, $\theta_i=(\mathrm{Tr}_1|v_i\rangle\langle v_i|)^*=A_i^TA_i^*$, $i=1,\ 2,\ \cdots,\ n$, where Tr_1 and Tr_2 stand for the traces over the first and second Hilbert spaces, respectively. Two "metric tensor" matrices $\Omega(\rho)$ and $\Theta(\rho)$ is given with entries

$$\Omega(\rho)_{ij}=\mathrm{Tr}(\rho_i\rho_j),\ \Theta(\rho)_{ij}=\mathrm{Tr}(\theta_i\theta_j),$$

for $i,\ j=1,\ 2,\ \cdots,\ n$, and

$$\Omega(\rho)_{ij}=\Theta(\rho)_{ij}=0,$$

for $n<i,\ j\leq d^2$.

Let $|\psi\rangle$ be a tripartite pure qudit in $\mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathcal{H}_3$. It can be regarded as a bipartite state by taking $\mathcal{H}_1\otimes\mathcal{H}_2$ and $\mathcal{H}_3$ as the two subsystems. Then denote the bipartite decomposition of $|\psi\rangle$ as $12-3$. Set a_{ijk} be the coefficients of $|\psi\rangle$ in orthonormal bases $|e_i\rangle\otimes|f_j\rangle\otimes|h_k\rangle$, here $|e_i\rangle,\ |f_i\rangle$ and $|h_i\rangle$ are orthonormal bases in Hilbert spaces $\mathcal{H}_1,\ \mathcal{H}_2$ and $\mathcal{H}_3$, respectively. Let $\mathcal{A}$ denote the matrix with respect to the bipartite state in $12-3$ decomposition, i.e. taking the subindices ij and k of a_{ijk} as the row and column indices of $\mathcal{A}$. Taking partial trace of $|\psi\rangle\langle\psi|$ over the third subsystem, we have $\sigma=\mathrm{Tr}_3|\psi\rangle\langle\psi|=\mathcal{A}\mathcal{A}^{\dagger}$. The reduced density matrix σ can be decomposed according to their eigenvalues μ_i and eigenvectors $|\xi_i\rangle$,

$$\sigma=\sum_{i=1}^{n}\mu_i|\xi_i\rangle\langle\xi_i|, \tag{3}$$

where $i=1,\ 2,\ \cdots,\ n$. Let $\mathcal{A}_i$ denote the matrix with entries given by the coefficients of $|\xi_i\rangle$ in the bases $|e_k\rangle\otimes|f_l\rangle$. We have $\rho_i'=\mathrm{Tr}_2|\xi_i\rangle\langle\xi_i|=\mathcal{A}_i\mathcal{A}_i^{\dagger}$, $\theta_i'=(\mathrm{Tr}_1|\xi_i\rangle\langle\xi_i|)^*=\mathcal{A}_i^T\mathcal{A}_i^*$, $i=1,\ 2,\ \cdots,\ n$. Denote $\Omega'(|\psi\rangle)$ and $\Theta'(|\psi\rangle)$ be two matrices with entries given by

$$\Omega'(|\psi\rangle)_{ij}=\mathrm{Tr}(\rho_i'\rho_j'),\ \Theta'(|\psi\rangle)_{ij}=\mathrm{Tr}(\theta_i'\theta_j'), \tag{4}$$

for $i,\ j=1,\ 2,\ \cdots,\ n$, and

$$\Omega'(|\psi\rangle)_{ij}=\Theta'(|\psi\rangle)_{ij}=0, \tag{5}$$

for $n<i,\ j\leq d^2$. Let $\mathcal{G}'$ be a class of tripartite pure states $|\psi\rangle$ satisfying

$$\mathrm{Det}(\Omega'(|\psi\rangle))\neq 0,\ \mathrm{Det}(\Theta'(|\psi\rangle))\neq 0. \tag{6}$$

Using the proposition, we can obtain a set of invariants for tripartite pure qudits in $\mathcal{G}'$:

$$\begin{aligned}
&J'^s(|\psi\rangle) = \mathrm{Tr}_2(\mathrm{Tr}_1 \rho'^s), \quad s = 1,\ 2,\ \cdots,\ d^2;\\
&\Omega'(|\psi\rangle)_{ij} = \mathrm{Tr}(\rho'_i \rho'_j),\ \Theta'(|\psi\rangle)_{ij} = \mathrm{Tr}(\theta'_i \theta'_j),\ i,\ j = 1,\ 2,\ \cdots,\ n;\\
&X'(|\psi\rangle)_{ijk} = \mathrm{Tr}(\rho'_i \rho'_j \rho'_k),\ Y'(|\psi\rangle)_{ijk} = \mathrm{Tr}(\theta'_i \theta'_j \theta'_k),\ i,\ j,\ k = 1,\ 2,\ \cdots,\ n.
\end{aligned} \tag{7}$$

Similarly, we can define the set of states $\mathcal{G}''$ ($\mathcal{G}'''$) via regarding $|\psi\rangle$ as a bipartite state in $\mathcal{H}_1$ ($\mathcal{H}_2$) and $\mathcal{H}_2 \otimes \mathcal{H}_3$ ($\mathcal{H}_1 \otimes \mathcal{H}_3$). The corresponding set of invariants for tripartite pure qudits in $\mathcal{G}''$ and $\mathcal{G}'''$ can be obtained in an analogous way.

It is well known that the invariants for two-qubit mixed states are studied and a complete set of 18 polynomial invariants is presented [6]. One can get the polynomial invariants of every three-qubit pure state $|\psi\rangle$ by using our proposition. Unlike the invariants obtained by Y. Makhlin in [6], the polynomial invariants can be represented by the coefficients of $|\psi\rangle$ under any bases and is convenient to calculate the invariants.

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